Int J Theor Phys (2010) 49: 1452–1456
DOI 10.1007/s10773-010-0325-2

Geometric Phase in the Interaction of a Time-Dependent Light Field with $\Im^{(3)}$ System

Qiang Liu · Zhao-Xian Yu · Zhi-Yong Jiao

Received: 22 January 2010 / Accepted: 18 March 2010 / Published online: 25 March 2010

Abstract By using of the invariant theory, we have studied the geometric phase in the interaction of a time-dependent light field with $\Im^{(3)}$ system, the dynamical and geometric phases are given, respectively. The disappearing condition of the geometric phase is given.

Keywords Geometric phase · $\Im^{(3)}$ system

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel and Bhandari [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric phases [11–14] in connection with the exact solutions of the corresponding time-dependent

Q. Liu · Z.-X. Yu (✉)
Department of Physics, Beijing Information Science and Technology University, Beijing 100192, China
e-mail: zxyu1965@163.com

Z.-Y. Jiao
Department of Physics, China University of Petroleum (East China), Dongying 257061, China

Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. In this paper, by using of the invariant theory, we shall study the geometric phase in the interaction of a time-dependent light field with $\Im^{(3)}$ system.

2 Model

The Hamiltonian of a time-dependent light field with $\Im^{(3)}$ system can be written by

$$\hat{H} = g(t)\hat{J}_3 + \nu(t)\hat{a}^\dagger\hat{a} + \lambda(t)[\hat{a}^\dagger\hat{J}_- + \hat{J}_+\hat{a}], \tag{1}$$

here $\nu(t)$ is the frequency of light field, and $\lambda(t)$ is the coupling coefficient between time-dependent light field and $\Im^{(3)}$ atom system. $\hat{J}_3$ and $\hat{J}_\pm$ are the generators of the $\Im^{(3)}$ algebra, and they satisfy the commutation relations

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = f\hat{J}_3 + h\hat{J}_3^3, \tag{2}$$

where the coefficient f is an arbitrary real number. In particular, when $f = 2$ or $f = -2$, and h is zero, then the $\Im^{(3)}$ algebra will reduce to the $SU(2)$ or $SU(1,1)$ algebras.

Introducing operators $\hat{P}_+ = \hat{a}^\dagger\hat{J}_-$ and $\hat{P}_- = \hat{J}_+\hat{a}$, one has

$$[\hat{P}_+, \hat{J}_3] = \hat{P}_+, \quad [\hat{P}_-, \hat{J}_3] = -\hat{P}_-, \quad [\hat{P}_+, \hat{a}^\dagger\hat{a}] = -\hat{P}_+, \quad [\hat{P}_-, \hat{a}^\dagger\hat{a}] = \hat{P}_-, \tag{3}$$

$$[\hat{P}_+, \hat{P}_-] = -f\hat{a}^\dagger\hat{a}\hat{J}_3 - h\hat{a}^\dagger\hat{a}\hat{J}_3^3 - \hat{J}_+\hat{J}_-, \quad [\hat{P}_+, \hat{J}_-] = [\hat{P}_-, \hat{J}_+] = 0. \tag{4}$$

In the following discussion, we consider the stronger coupling case between time-dependent light field and the $\Im^{(3)}$ system, so we can let $\lambda_{NJ} = -\langle f\hat{a}^\dagger\hat{a}\hat{J}_3 + \hat{J}_+\hat{J}_-\rangle$, and $H = -\langle h\hat{J}_3^3\rangle$, so one has $[\hat{P}_+, \hat{P}_-] = \lambda_{NJ} + H\hat{a}^\dagger\hat{a}$. Equation (1) can be rewritten by

$$\hat{H} = g(t)\hat{J}_3 + \nu(t)\hat{a}^\dagger\hat{a} + \lambda(t)[\hat{P}_+ + \hat{P}_-]. \tag{5}$$

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial\hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{6}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{7}$$

where $\frac{\partial\lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{8}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (8) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$ for the non-degenerate state, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{9}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (8). Then the general solution of the Schrödinger equation (8) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{10}$$

where

$$\delta_n(t) = \int_0^t dt' \langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{11}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

We can introduce the L-R invariant as follows

$$\hat{I} = \alpha(t)\hat{P}_+ + \alpha^*(t)\hat{P}_- + \beta(t)\hat{J}_3. \tag{12}$$

Substituting (5) and (12) into (6), one has the auxiliary equations

$$i\dot{\alpha}(t) + \alpha(t)[g(t) - \nu(t)] - \beta\lambda(t) = 0, \quad \beta = \text{const}, \ \alpha(t) = \alpha^*(t), \tag{13}$$

where dot denotes the time derivative.

We now construct the unitary transformation

$$\hat{V}(t) = \exp[\xi(t)\hat{P}_+ - \xi^*(t)\hat{P}_-], \tag{14}$$

it is easy to find that when satisfying the following relations

$$\frac{\beta[1-\cos(\sqrt{2H}|\xi(t)|)]}{\sqrt{2H}|\xi(t)|} - \frac{\alpha(t)\sqrt{H}[\xi(t)+\xi^*(t)]}{\sqrt{2}|\xi(t)|}\sin(\sqrt{2H}|\xi(t)|) = 1, \quad \beta = 1, \tag{15}$$

$$\frac{\alpha(t)}{2}[1+\cos(\sqrt{2H}|\xi(t)|)] - \frac{\alpha(t)\xi^2(t)}{2|\xi(t)|^2}[1-\cos(\sqrt{2H}|\xi(t)|)] - \frac{\xi(t)\sin(\sqrt{2H}|\xi(t)|)}{\sqrt{2H}|\xi(t)|} = 0, \tag{16}$$

$$\frac{\alpha(t)[\xi(t)+\xi^*(t)]}{\sqrt{2H}|\xi(t)|}\sin(\sqrt{2H}|\xi(t)|)] - \frac{1}{H}[1-\cos(\sqrt{2H}|\xi(t)|)] = 0, \tag{17}$$

then a time-independent invariant $\hat{I}_V$ appears

$$\hat{I}_V = \hat{V}^\dagger(t)\hat{I}\hat{V}(t) = \hat{J}_3. \tag{18}$$

In terms of the unitary transformation $\hat{V}(t)$ and the Baker-Campbell-Hausdoff formula [28]

$$\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t} = \frac{\partial\hat{\wp}}{\partial t} + \frac{1}{2!}\left[\frac{\partial\hat{\wp}}{\partial t}, \hat{\wp}\right] + \frac{1}{3!}\left[\left[\frac{\partial\hat{\wp}}{\partial t}, \hat{\wp}\right], \hat{\wp}\right] + \frac{1}{4!}\left[\left[\left[\frac{\partial\hat{\wp}}{\partial t}, \hat{\wp}\right], \hat{\wp}\right], \hat{\wp}\right] + \cdots, \tag{19}$$

where $\hat{V}(t)=\exp[\wp(t)]$. It is easy to find that when satisfying the following relation

$$\begin{aligned}&\frac{[\nu(t)-g(t)]\xi(t)}{\sqrt{2H}|\xi(t)|}\sin(\sqrt{2H}|\xi(t)|)+\frac{\lambda(t)}{2}[1+\cos(\sqrt{2H}|\xi(t)|)]\\&\quad-\frac{\lambda(t)\xi^2(t)}{2|\xi(t)|^2}[1-\cos(\sqrt{2H}|\xi(t)|)]-i\dot{\xi}(t)\\&\quad+\frac{i\xi(t)[\xi(t)\dot{\xi}^*(t)-\dot{\xi}(t)\xi^*(t)]}{2\sqrt{2H}|\xi(t)|^3}[\sin(\sqrt{2H}|\xi(t)|)-\sqrt{2H}|\xi(t)|]=0,\end{aligned}\tag{20}$$

one has

$$\begin{aligned}\hat{H}_V(t)&=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}\\&=\nu(t)\cos(\sqrt{2H}|\xi(t)|)\hat{a}^\dagger\hat{a}+g(t)\hat{J}_3\\&\quad+\frac{g(t)}{\sqrt{2H}|\xi(t)|}[1-\cos(\sqrt{2H}|\xi(t)|)]\hat{a}^\dagger\hat{a}\\&\quad+\frac{\lambda_{NJ}[g(t)-\nu(t)]}{H}[1-\cos(\sqrt{2H}|\xi(t)|)]\\&\quad-\frac{\lambda_{NJ}\lambda(t)[\xi(t)+\xi^*(t)]}{\sqrt{2H}|\xi(t)|}\sin(\sqrt{2H}|\xi(t)|)\\&\quad-\frac{\lambda(t)\sqrt{H}[\xi(t)+\xi^*(t)]}{\sqrt{2}|\xi(t)|}\sin(\sqrt{2H}|\xi(t)|)\hat{a}^\dagger\hat{a}\\&\quad-\frac{i[\xi(t)\dot{\xi}^*(t)-\dot{\xi}(t)\xi^*(t)]}{2|\xi(t)|^2}[1-\cos(\sqrt{2H}|\xi(t)|)]\hat{a}^\dagger\hat{a}\\&\quad-\frac{i\lambda_{NJ}[\xi(t)\dot{\xi}^*(t)-\dot{\xi}(t)\xi^*(t)]}{2H|\xi(t)|^2}[1-\cos(\sqrt{2H}|\xi(t)|)].\end{aligned}\tag{21}$$

The phase $\delta(t)=\delta^d(t)+\delta^g(t)$ includes the dynamical phase

$$\begin{aligned}\delta^d(t)=&-\int_{t_0}^{t}[n\nu(t')\cos(\sqrt{2H}|\xi(t')|)+mg(t')]dt'\\&-\int_{t_0}^{t}\frac{1}{\sqrt{H}}\left\{\frac{ng(t')}{\sqrt{2}|\xi(t')|}+\frac{\lambda(t')[g(t')-\nu(t')]}{\sqrt{H}}\right\}[1-\cos(\sqrt{2H}|\xi(t')|)]dt'\\&+\int_{t_0}^{t}\frac{\lambda(t')[\xi(t')+\xi^*(t')]}{\sqrt{2}|\xi(t')|}\left[\frac{\lambda_{NJ}}{\sqrt{H}}+n\sqrt{H}\right]\sin(\sqrt{2H}|\xi(t')|)dt',\end{aligned}\tag{22}$$

and the geometric phase

$$\delta^g(t)=i\int_{t_0}^{t}\frac{[\xi(t')\dot{\xi}^*(t')-\dot{\xi}(t')\xi^*(t')]}{2|\xi(t')|^2}[1-\cos(\sqrt{2H}|\xi(t')|)]\left[n+\frac{\lambda_{NJ}}{H}\right]dt',\tag{23}$$

where we have used relations $\hat{a}^\dagger\hat{a}|n\rangle=n|n\rangle$ and $\hat{J}_3|J,m\rangle=m|J,m\rangle$. In particular, when $n+\frac{\lambda_{NJ}}{H}=0$, the geometric phase will disappear.

References

1. Pancharatnam, S.: Proc. Indian Acad. Sci. A **44**, 247 (1956)
2. Berry, M.V.: Proc. R. Soc. Lond. A **392**, 45 (1984)
3. Aharonov, Y., Anandan, J.: Phys. Rev. Lett. **58**, 1593 (1987)
4. Samuel, J., Bhandari, R.: Phys. Rev. Lett. **60**, 2339 (1988)
5. Mukunda, N., Simon, R.: Ann. Phys. (N.Y.) **228**, 205 (1993)
6. Pati, A.K.: Phys. Rev. A **52**, 2576 (1995)
7. Uhlmann, A.: Rep. Math. Phys. **24**, 229 (1986)
8. Sjöqvist, E.: Phys. Rev. Lett. **85**, 2845 (2000)
9. Tong, D.M., et al.: Phys. Rev. Lett. **93**, 080405 (2004)
10. Lewis, H.R., Riesenfeld, W.B.: J. Math. Phys. **10**, 1458 (1969)
11. Gao, X.C., Xu, J.B., Qian, T.Z.: Phys. Rev. A **44**, 7016 (1991)
12. Gao, X.C., Fu, J., Shen, J.Q.: Eur. Phys. J. C **13**, 527 (2000)
13. Gao, X.C., Gao, J., Qian, T.Z., Xu, J.B.: Phys. Rev. D **53**, 4374 (1996)
14. Shen, J.Q., Zhu, H.Y.: arXiv:quant-ph/0305057v2 (2003)
15. Richardson, D.J., et al.: Phys. Rev. Lett. **61**, 2030 (1988)
16. Wilczek, F., Zee, A.: Phys. Rev. Lett. **25**, 2111 (1984)
17. Moody, J., et al.: Phys. Rev. Lett. **56**, 893 (1986)
18. Sun, C.P.: Phys. Rev. D **41**, 1349 (1990)
19. Sun, C.P.: Phys. Rev. A **48**, 393 (1993)
20. Sun, C.P.: Phys. Rev. D **38**, 298 (1988)
21. Sun, C.P., et al.: J. Phys. A **21**, 1595 (1988)
22. Sun, C.P., et al.: Phys. Rev. A **63**, 012111 (2001)
23. Chen, G., Li, J.Q., Liang, J.Q.: Phys. Rev. A **74**, 054101 (2006)
24. Chen, Z.D., Liang, J.Q., Shen, S.Q., Xie, W.F.: Phys. Rev. A **69**, 023611 (2004)
25. He, P.B., Sun, Q., Li, P., Shen, S.Q., Liu, W.M.: Phys. Rev. A **76**, 043618 (2007)
26. Li, Z.D., Li, Q.Y., Li, L., Liu, W.M.: Phys. Rev. E **76**, 026605 (2007)
27. Niu, Q., Wang, X.D., Kleinman, L., Liu, W.M., Nicholson, D.M.C., Stocks, G.M.: Phys. Rev. Lett. **83**, 207 (1999)
28. Wei, J., Norman, E.: J. Math. Phys. **4**, 575 (1963)